MATH 1910: Exam 2 Bonus   Name: UUID:

Each question will be graded based on effort alone. Feel free to use the textbook or material from other sources to investigate these questions. I also don't mind if you work together on it. For each problem you can get either 0 or 2 points added to your exam score. The only requirement to score 2 points on a single problem is that you show me that you made an effort to think about it. So you can get as much as 10 points added to your gross exam score. Since it was a 100 point exam this corresponds to a whole letter grade.

Due: October 28th

1. Find $\frac{d}{dx}\log_x(2)$ where $x > 0$ and $x \neq 1$. Try the change of base formula or implicit differentiation.

2. (Polynomial interpolation) Given three points in the plane, $(x_0, y_0), (x_1, y_1)$, and (x_2, y_2) we can produce a polynomial which passes through all three points using polynomial interpolation. (We can do this with any finite number of points. Here I only do it with 3 points for simplicity.) The interpolating polynomial looks like:

$$p(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

   This specific method of interpolation is named after Lagrange and this polynomial is called the Lagrange polynomial. Verify that $p(x_0) = y_0$, $p(x_1) = y_1$ and $p(x_2) = y_2$. Find the interpolating polynomial for the points $\left(1/2, 1/\sqrt{3}\right)$, $(1,1)$, and $\left(3/2, \sqrt{3}\right)$. Fully expand and simplify. Verify that it goes through the given points.

3. ("Quadratic" approximation) We may approximate a function using polynomial interpolation. Given three points that are near each other, $x_0 - h$, x_0, $x_0 + h$, we can estimate our function f near x_0 by using polynomial interpolation with the points $(x_0 - h, f(x_0 - h))$, $(x_0, f(x_0))$, and $(x_0 + h, f(x_0 + h))$. This looks like the following:

$$\begin{aligned} f(x) \approx{} & f(x_0 - h)\frac{(x-x_0)(x-x_0-h)}{2h^2} \\ & + f(x_0)\frac{(x-x_0+h)(x-x_0-h)}{-h^2} \\ & + f(x_0+h)\frac{(x-x_0+h)(x-x_0)}{2h^2}. \end{aligned}$$

Differentiating the polynomial gives an estimate of f' near x_0. Do you think it would be a good estimate? Use these methods to estimate

$$f(x) = \arctan\left(\frac{8-4\sqrt{3}}{\sqrt{3}}x^2 + \frac{-14+8\sqrt{3}}{\sqrt{3}}x + \frac{6-3\sqrt{3}}{\sqrt{3}}\right)$$

using $x_0 = 1$ and $h = 1/2$. Notice, $f(1/2) = \pi/6$, $f(1) = \pi/4$, and $f(3/2) = \pi/3$. Plot the function and its estimate. (The link for it in desmos: here.) Simplify the estimate of f then differentiate. Do you think this is close to the derivative of f near 1? Fully differentiate f and find its value at 1. How close is the approximation of the derivative to the actual value?

4. Two curves are orthogonal if they are orthogonal at ever point of their intersection. Show that

$$x^2 + y^2 = x \text{ and } x^2 + y^2 = y$$

are orthogonal. Plot the solutions to the two equations. Find the points at which they intersect. Then implicitly differentiate to check that the slopes of their tangent lines are orthogonal at the points of their intersection.

5. (*e*) Let's find $\lim_{x\to 0}(x+1)^{1/x}$ using log rules and the limit definition of the derivative. Notice that $\ln(x)$ is continous at $(x+1)^{1/x}$ for all $x > 0$. Thus $\ln\left(\lim_{x\to 0}(x+1)^{1/x}\right) = \lim_{x\to 0}\ln\left((x+1)^{1/x}\right)$. Let $L = \lim_{x\to 0}(x+1)^{1/x}$. Then $\ln L = \lim_{x\to 0}\ln\left((x+1)^{1/x}\right)$. Calculate the limit on the right hand side by interpreting it as the limit definition of the derivative of $\ln(x)$ at 1. Then solve for L.