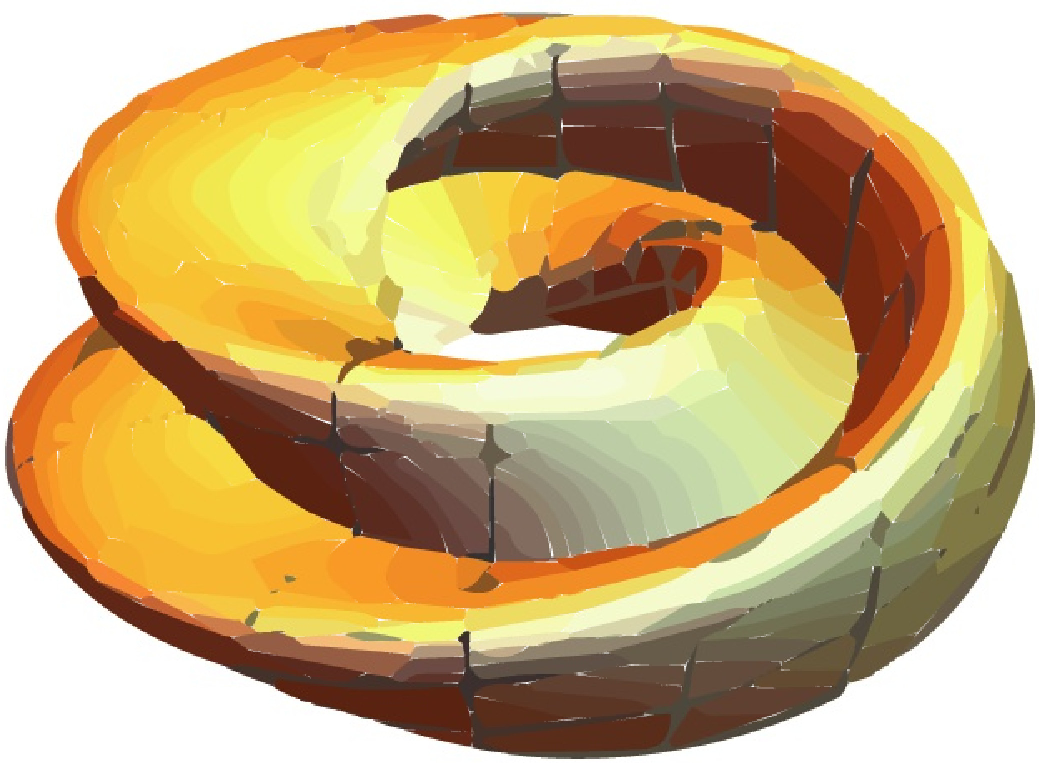
CONNER GRIFFIN

MULTIVARIABLE CALCULUS



Contents

Chapter 1

Euclidean Space

1.1 The Cartesian Product

Euclidean space is very important to the study of multivariable calculus. Understanding the algebra and geometry of the space and of the various maps we have between euclidean spaces of different dimensions is the key to unlocking the material in these notes. To understand euclidean space we first need to make some notes about the cartesian product. The cartesian product should be something you are fairly familiar with already, even if you do not know it by name. It is the product which gives us the euclidean plane, that is the set of all ordered pairs of real numbers. This is where we visualize the graphs of the functions you know and love from single variable calculus. More generally when we say cartesian product this is what we mean:

Definition 1.1.1 (Cartesian Product)**.** *The cartesian product of two sets, A and B, is the set of all ordered pairs with elements (a, b) where a is in A and b is in B. We denote the cartesian product as $A \times B$.*[a]

[a]Note that $A \times B = B \times A$ only when $A = B$.

This definition is for sets in general. In our case, the graphs of our single variable functions are in the euclidean plane, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.[1] Points on our graph look like $(x, f(x))$ which is indeed an ordered pair of real numbers. In fact, for a given function f, which has domain some set A and range some set B, the graph of f is a function from A to $A \times B$.

[1]We use $\mathbb{R}$ to denote the real line.

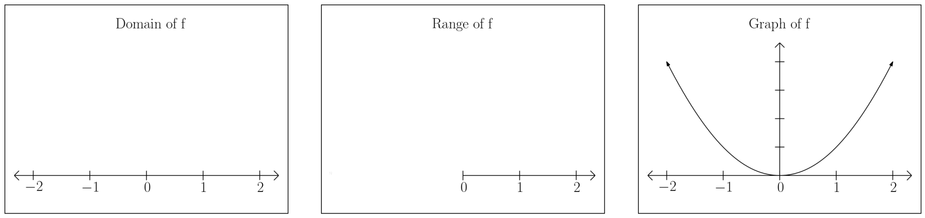


Figure 1.1: Domain, range and graph of f

Example 1.1.1. Consider the function $f(x) = x^2$. The domain of the function is $\mathbb{R}$, the range is $[0, \infty)$, but the graph of f, which is all of the points $(x, f(x))$, is contained in $\mathbb{R}^2$.

It is important to note that the cartesian product is in some sense associative. By this we mean that

$$(A \times B) \times C = A \times (B \times C)$$

where A, B, and C are non empty sets and the parentheses indicate that that is the product you take first. So, elements of $(A \times B) \times C$ look like $((a, b), c)$ and elements of $A \times (B \times C)$ look like $(a, (b, c))$. We equate these two products by the following identification $((x, y), z) = (x, (y, z))$. This is not equality in the sense of two real numbers being equal. This is an invertible assignment that preserves the algebraic structure of $(A \times B) \times C$ and $A \times (B \times C)$. This is a type of equality in its own way. Associativity is an important quality of our operation because it allows us to take multiple products without specifying the order. As such, we may express both $(A \times B) \times C$ and $A \times (B \times C)$ as $A \times B \times C$.

Definition 1.1.2 (Multiple Cartesian Products)**.** *The caresian product of n sets*[a]*, $A_1, A_2, \ldots A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_1 is an element of A_1, a_2 is an element of A_2 et cetera. We denote the cartesian product in three main ways, $\prod_{i=1}^{n} A_i = \times_{i=1}^{n} A_i = A_1 \times A_2 \times \cdots \times A_n$.*[b]

[a] Here, n is a positive integer and represents the total number of sets of which we will be taking the cartesian product

[b] Think of the first two as mirroring summation notation as in $\sum_{i=1}^{n} a_n$.

Definition 1.1.3 (Euclidean Space)**.** *The n-dimensional euclidean space is the multiple cartesian product of n copies of the real line, $\times_{i=1}^{n} \mathbb{R}$. We denote it as $\mathbb{R}^n$.*

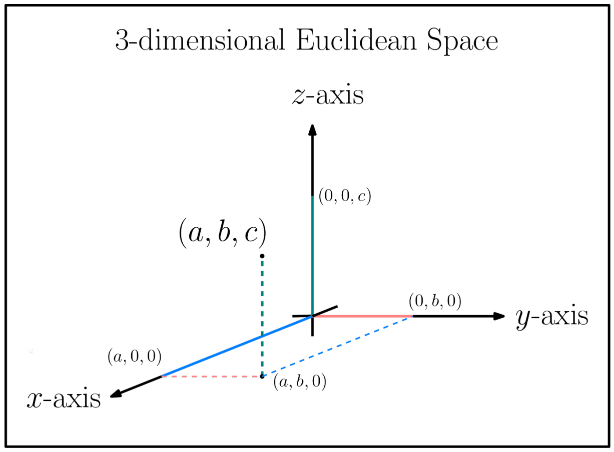


Figure 1.2: A point, (a, b, c), in $\mathbb{R}^3$

For multivariable calculus, euclidean space is the place where everything lives. It will contain the domain, range and graphs of the functions we will be considering. The most important of these to familiarize yourself with is $\mathbb{R}^3$ which we may visualize as in Figure 1.2.

Example 1.1.2. Consider the function $f(x) = (x, x^2)$. The domain of f is $\mathbb{R}$, the range is $\mathbb{R} \times [0, \infty)$, and the graph of the function is all of the points $(x, (x, x^2))$. We equate this point, which is in $\mathbb{R} \times (\mathbb{R} \times [0, \infty))$, with the point (x, x, x^2) in $\mathbb{R} \times \mathbb{R} \times [0, \infty)$.

Example 1.1.3. Another familiar example of a function from $\mathbb{R}$ to $\mathbb{R}^2$ is the unite circle: $f(\theta) = (\cos(\theta), \sin(\theta))$.

1.2 Algebra and Geometry of Euclidean Space

It is because of the algebra of the space that we refer to elements of $\mathbb{R}^n$ as *vectors*.

1.2.1 Vector Addition and Scalar Multiplication in $\mathbb{R}^n$.

$(a, b) + (x, y)$

1.2.2 The Dot Product

1.2.3 Algebra and Geometry of $\mathbb{R}^3$

Chapter 2

Functions from $\mathbb{R}$ to $\mathbb{R}^2$

2.1 Limits and Continuity

2.2 Differentiation

2.3 Integration

2.4 Functions from $\mathbb{R}$ to $\mathbb{R}^3$

2.5 Arclength and Curvature

Chapter 3

Functions from $\mathbb{R}^2$ to $\mathbb{R}$

Chapter 4

Functions from $\mathbb{R}^2$ to $\mathbb{R}^2$