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Differential Equations

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These notes will provide a brief overview of the material we cover in Math 2120. For additional worked problems, I will often link to Paul's Online Math Notes. I recommend that you keep up with the material by working through the relevant sections there.

Contents

1 Introduction to Differential Equations **1**
1.1 Algebraic equations 1
1.2 Differential equations 2
1.3 Intro to Differential Equations Lecture Notes 4

2 First order equations **11**
2.1 Seperable equations 11
2.2 Seperable Equations Lecture Notes 13
2.3 First order linear equations 26
2.4 Linear Equations Lecture Notes 27
2.5 Exact equations 32
2.6 Exact Equations Lecture Notes 34

Chapter 1

Introduction to Differential Equations

As a subject, differential equations is an extension of calculus which is necessary for solving many problems in engineering and the physical sciences. Due to its applicability, it is a required course for math, physics, and engineering majors and is recommended for some concentrations of biology, economics and computer science. The extension of the subject to include multivariate functions – hence partial derivatives – is currently one of the most active areas of mathematical research.

1.1 Algebraic equations

This course is devoted entirely to a new kind of mathematical object: differential equations. As a gentle introduction we will compare them to algebraic equations which should be familiar to you.

Definition 1.1.1 (Algebraic expression)**.** *An* algebraic expression *is a mathematical expression composed of real numbers, variables and the algebraic operations (addition, multiplication, exponentiation, etc.) in such a way that if real numbers are substituted for the variables then the resulting expression would be a real number.*

Example 1.1.2. The following are all examples of algebraic expressions.

$$x^2 + y^2, mx + b, 12(13)$$

Definition 1.1.3 (Algebraic equation)**.** *An* algebraic equation *is a pair of algebraic expressions related to each other by equality.*

Example 1.1.4. The following are all examples of algebraic equations.

$$x^2 + y^2 = 1, y = mx + b, 2a = 12(13)$$

Definition 1.1.5. *A* solution *to an equation with variables $x_1, x_2, \ldots, x_n$ is an ordered list of n real numbers such that if for all $1 \leq i \leq n$, the ith real number of the solution is substituted for x_i then the equality is true.*

Example 1.1.6. The ordered pair $\left(1, \sqrt{2}\right)$ is a solution to the equation

$$x + y^2 = 3.$$

A problem like the following is very common in any algebra class.

Example 1.1.7. Find x given that $\sqrt{1-x^2} = x$.
Solution:

$$\begin{aligned} \sqrt{1-x^2} = x &\Rightarrow 1 - x^2 = x^2 \quad \text{note we have possibly added solutions} \\ &\Rightarrow 1 = 2x^2 \\ &\Rightarrow \frac{1}{2} = x^2 \\ &\Rightarrow x = \pm\frac{1}{\sqrt{2}} \end{aligned}$$

Going back and checking the original equation we see that only $x = \frac{1}{\sqrt{2}}$ can be a solution.

1.2 Differential equations

A differential equation is an equation involving an unknown function or functions (also called dependent variables) as well as operations on that function, including especially the derivative. Typically in this course y will be the dependent variable and x (sometimes t) the independent variable. That is y is a function in the variable x. The GOAL of differential equations is to find a differentiable function, f, which solves the equation on some interval. That is when f is plugged in for y in the equation the equality holds true for all x-values in some interval.

Example 1.2.1. The following is a differential equation that we will learn how to solve in this course.

$$3y' + xy = 0$$

One solution is $f(x) = e^{-x^2/6}$ with the interval of solution being the entire real line. Indeed, $f'(x) = \frac{-x}{3}e^{-x^2/6}$ and then

$$3f'(x) + xf(x) = 3\frac{-x}{3}e^{-x^2/6} + xe^{-x^2/6} = 0$$

for all real numbers x.

We will learn how to solve several classes of differential equations. There are two main ways that we classify DEs.

1. Order

   The order of a DE is the order of the highest derivative in the equation.

   $$\text{Ex: } y'' + 5y' - 4y = e^x \text{ is a second order DE.}$$

2. Linearity

   An nth order DE can be expressed in the form

   $$F\left(x, y, y', \ldots, y^{(n)}\right) = 0$$

   for some function $F : \mathbb{R}^{n+1} \to \mathbb{R}$.

   An nth order DE is linear if F is linear in $y, y', \ldots, y^{(n)}$. More specifically it is linear if it looks like

   $$a_n(x)\, y^{(n)} + \cdots + a_1(x)\, y' + a_0(x)\, y + f(x) = 0.$$

   $$\text{Ex: } x^3 \frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = \ln x \text{ is a third order linear DE.}$$

   $$\text{Ex: } \frac{d^4y}{dx^4} + \left(\frac{dy}{dx}\right)^2 = 0 \text{ is a fourth order nonlinear DE.}$$

The general idea behind solving differential equations is not very different from algebraic equations. We have many operations that we can apply to functions: addition, multiplication, exponentation, trig functions, inverses of all these, function composition and (now that we've had calculus) *differentiation* as well as it's inverse *integration*. Thus to accomplish the goal of differential equations (finding a function or functions that satisfy the equation) we aim to isolate the dependent variable y by applying inverse operations including especially *integration*.

Introduction To Diff Eq

In any algebra course, finding solutions to equations is a major component.

Ex: Find all x such that

$$x^2 - 3x + 1 = 0.$$

Soln: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9-4}}{2}$$

$$= \frac{3+\sqrt{5}}{2}, \quad \frac{3-\sqrt{5}}{2}$$

Ex: Find x and y such that

$$\begin{cases} 3x + y = 2 \\ x - y = 1. \end{cases}$$

Soln: $x - y = 1 \rightarrow y = x - 1$

$3x + y = 2 \rightarrow 3x + (x-1) = 2$

$\rightarrow 4x - 1 = 2$

$\rightarrow x = \frac{3}{4}$

$\rightarrow y = \frac{3}{4} - 1 = -\frac{1}{4}$

Ex: Find all x such that

$$x^4 - 5x^2 + 4 = 0$$

Soln: Let $u = x^2$

Then $x^4 - 5x^2 + 4 = (x^2)^2 - 5x^2 + 4$

$$= u^2 - 5u + 4$$

$$= (u-4)(u-1) = 0$$

$$x^2 = u = 4, 1 \rightarrow x = \pm 2, \pm 1$$

In all of these examples inverse operations were used to "isolate" x.

In calculus two new operations on functions are introduced

1) Differentiation
2) Integration

The Fundamental Theorem of Calculus establishes that these two operations are kind of inverses.

That is

$$\int_a^x f'(t)\,dt = f(x) + C \quad \text{for some real number } C.$$

and $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

This connection between diff and integration will allow us to solve differential equations which are equations involving an unknown function, y, (or functions) and operations on that function including differentiation.

Ex: We will learn how to solve the following differential equations.

$$3y' + xy = 0, \qquad \begin{cases} y_1' + 3y_1 - y_2' = 2 \\ y_1' + y_2' - 2y_2 = 1 \end{cases}$$

or $x^2 y'' - xy' + y = \ln(x)$

Let's solve our first differential equation of the semester. You will have seen an example like this in calculus.

Example: Suppose $y' = 3x^2 - x + 1$ and $y(0) = 1$.

Soln: $y(x) = \int y'\,dx = \int 3x^2 - x + 1\,dx = x^3 - \frac{1}{2}x^2 + x + C$

$$y(0) = 0^3 - \frac{1}{2}0^2 + 0 + C = C = 1$$

Thus $y(x) = x^3 - \frac{1}{2}x^2 + x + 1$

This is a very simple differential equation. We will see that

more complicated equations will require advanced solution methods.

A major skill that we will be developing over the semester is the ability to identify different classes of differential equations

→ Classification by order

The order of a differential equation is the order of the highest derivative in the equation.

Ex: $y'' + 5(y')^3 - 4y = e^t$

is a second-order diff eq.

$$4x\frac{dy}{dx} + y = x$$

is a first-order diff eq.

In the case of 1st-order we will occasionally write

$$4x\,dy + (y-x)dx = 0$$ instead

as if we have multiplied by dx.

We can express any nth order differential equation in the general form

$$F(x, y, y', \cdots, y^{(n)}) = 0.$$

→ Classification by linearity

An nth order differential equation

if F is linear in $y, y', \cdots, y^{(n)}$

More specifically it is linear if it looks like

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y + g(x) = 0$$

Examples: (1) $(y-x)dx + 4x\,dy = 0$

(2) $y'' - 2y + y = 0$

(3) $x^3\frac{d^3 y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$

(1) Linear first-order

(2) Linear second-order

(3) Linear third-order

Examples (of nonlinear de's):

(1) $(1-y)y' + 2y = e^x$

This is nonlinear because the coefficient of y', $1-y$, is dependent on y.

(2) $\frac{d^2y}{dx^2} + \sin(y) = 0$

This is nonlinear because $\sin(y)$ is a nonlinear function in y.

(3) $\frac{d^4y}{dx^4} + y^2 = 0$

Nonlinear because y^2 is nonlinear function in y.

Chapter 2

First order equations

Up to the first exam we will only be concerned with first order equations. That is differential equations whose highest order derivative is the first derivative. There will be many classes of first order equations that we will cover: seperable equations, linear first order equations, exact equations, homogeneous first order equations, Bernoulli and substitution equations. Some of these classes overlap. The final section of this chapter will cover modeling with first order equations.

An arbitrary first order equation looks like

$$M(x,y)\frac{dy}{dx}+N(x,y)=0.$$

Where M, N are continuous functions.

Which we will often write as

$$M(x,y)\,dy+N(x,y)\,dx=0.$$

2.1 Seperable equations

Seperable differential equations are first order differential equations that can be written in the form

$$f(y)\frac{dy}{dx}=g(x)$$

for some continuous functions f, g.

The method of solution is rather direct and relies on the chain rule. Note that the chain rule says that $\int f(y)\,y'dx=\int f(y)\,dy$.

Solution:

$$f(y)\frac{dy}{dx}=g(x)\Rightarrow\int f(y)\,y'dx=\int g(x)\,dx+C\quad\text{(integrate both sides w.r.t) } x$$
$$\Rightarrow F(y)=\int g(x)\,dx+C\quad\text{where } F \text{ is an antiderivative of } f$$

This is as far as we can take things in general. For many examples, we will be able to solve for y as a function in x. You should solve for y when possible.

Of course, many separable equations that you will be asked to solve will not already be in the general form $f(y)\frac{dy}{dx} = g(x)$. Therefore, you will be expected to first rearrange the equation so that it is in the general form so that you can then apply the above method of solution.

Paul's link: Seperable equations

SEPARABLE EQUATIONS

The first class of differential equations we will learn to solve is a sub-class of 1st order equations, separable equations.

DEF: (Separable Equation)

A first-order differential equation of the form

$$h(y)\frac{dy}{dx} = g(x)$$

is said to be separable or to have separable variables.

Integration plays a major role in solving these equations and we will have to employ many of the major integration techniques.

IT IS RECOMENDED THAT YOU REFRESH YOURSELF ON U-SUB, INTEGRATION BY PARTS AND PARTIAL FRACTION DECOMP BEFORE EXAM 1.

ID QUIZ

Which of the following DEs are separable?

(a) $\sin(y)\frac{dy}{dx} = x^2$

(b) $\frac{dy}{dx} = y^2 x e^{3x+y}$

(c) $\frac{dy}{dx} = x^2 y + 1$

(d) $(y^2+y+1)\frac{d^2y}{dx^2} = \tan(x)$

Soln: (a) Y (b) Y (c) N (d) N

Method of Solution

Observe, if we integrate both sides with respect to x,

$$\int h(y)y'\,dx = \int g(x)\,dx,$$ then

the method of u-sub allows us to treat the left hand side as $\int h(y)\,dy$.

Suppose $H(y)$ and $G(x)$ are antiderivatives

of $h(y)$ and $g(x)$, respectively. Then

$$\int h(y)\,dy = \int g(x)\,dx \quad \text{becomes}$$

$$H(y) = G(x) + C^{*} \quad \text{for some constant } C.$$

whether or not we can solve for y as a function in x depends on the invertibility of H, that is whether or not H can be undone. Possibly it is only invertible on part of its domain. In either case we have a solution curve. We will see an example where H is not wholly invertible and discuss how one interprets the soln in this case.

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* We've antidifferentiated twice but there is no need for two constants – as in $H(y) + C_1 = G(x) + C_2$ — because $C_2 - C_1$ is a perfectly fine representation for any real number and hence can be replaced by a single constant C.

An introductory example

Solve $(1+x)dy - ydx = 0$.

Soln: Remember that this is another way of writing

$$(1+x)\frac{dy}{dx} - y = 0$$

Step 1: Get all of the y variables on the same side as dy/dx.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{1+x}$$

Step 2: Integrate both sides w.r.t x.

$$\int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{1}{1+x}dx \quad (*)$$

$$LHS \overset{y\text{-sub}}{=} \int \frac{1}{y}dy = \ln|y| + C$$

$$RHS = \int \frac{1}{u}du = \ln|u| = \ln(1+x) + C$$

where $u = 1+x$

(*) Recall that substitution is the chain rule backwards

Chain Rule

$$\frac{d}{dx}f(y) = f'(y)y'$$

Substitution

$$\int f'(y)y'dx = f(y) + C$$

we sort of antiff f w.r.t y

Step 3: When possible solve for y.

$$\ln|y| = \ln|1+x| + C$$

→ the inverse of $\ln(y)$ is e^y

So we apply it to both sides

$$e^{\ln|y|} = e^{\ln|1+x|+C}$$

$$|y| = e^C\,|1+x|$$

$$\left|\frac{y}{1+x}\right| = e^C$$

$$\frac{y}{1+x} = \pm e^C = c \neq 0$$

Hence $y = c(1+x)$

Let's double check our solution.

$$\frac{dy}{dx} = c = \frac{c(1+x)}{1+x} = \frac{y}{1+x}$$

By observation $y=0$ is a solution which was "lost" when we divided both sides by y.

Note: constant solutions to DEs are also called equilibrium solutions.

EX: Solve $\frac{dy}{dx} = \frac{-x}{y}$, $y(4) = -3$

→ This is an initial value problem because we are given the initial value $y(4)=-3$.

→ Do you remember implicit differentiation in calc 1? Almost always the first example you see is finding $\frac{dy}{dx}$ when $x^2+y^2=1$

↳ $2x+2yy'=0$

↳ $y' = \frac{-x}{y}$

Soln: $\frac{dy}{dx} = \frac{-x}{y}$

$y\frac{dy}{dx} = -x$

$\int y\frac{dy}{dx}dx = \int -x\,dx$

$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$

$y^2+x^2 = 2C = r^2$, we now apply our initial value condition

$(y(4))^2 + 4^2 = r^2$

$(-3)^2 + 16 = r^2$

$9 + 16 = 25 = r^2$

$y^2 + x^2 = 5^2$, a circle of radius 5

and hence y cannot be expressed as a function in x without losing some solutions.

Because of the simplicity of the equation we can determine some explicit solutions from the implicit solution ($y^2 + x^2 = 25$)

We have

$y = \sqrt{25 - x^2}$ and

$y = -\sqrt{25 - x^2}$ as our two

options for explicit solutions.

the initial value satisfies only

$y = -\sqrt{25 - x^2}$

IN-CLASS EXERCISES

1) Solve $\frac{dy}{dx} = y^2 - 4$

2) Solve $(e^{2y} - y)\cos(x)\frac{dy}{dx} = e^y \sin(2x)$, $y(0) = 0$

Soln (1):

$$\frac{1}{y^2-4} \frac{dy}{dx} = 1 \quad *$$

* by dividing by y^2-4 we are losing solns $y = \pm 2$.

$$\int \frac{1}{y^2-4}\, dy = \int 1\, dx = x + c$$

partial fractions

$$\frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2} \rightarrow 1 = (y+2)A + (y-2)B$$

$$\rightarrow \begin{cases} A + B = 0 \\ 2A - 2B = 1 \end{cases}$$

$$\rightarrow 4A = 1 \rightarrow A = \tfrac{1}{4}, B = -\tfrac{1}{4}$$

$$\int \frac{1}{y^2-4}\, dy = \int \frac{1/4}{y-2} + \frac{-1/4}{y+2}\, dy$$

$$= \frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2|$$

$$= \ln\left|\frac{(y-2)^{1/4}}{(y+2)^{1/4}}\right|$$

$$\ln\left|\frac{(y-2)^{1/4}}{(y+2)^{1/4}}\right| = x + C$$

$$\left(\frac{y-2}{y+2}\right)^{1/4} = Ce^{x} \qquad C > 0$$

why $C > 0$? because $e^{C} > 0$ for any real # C

$$\frac{y-2}{y+2} = Ce^{4x} \qquad C \neq 0$$

why $C \neq 0$? when we take 4th

$$y-2 = (y+2)C$$

This direction loses solutions!

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x + C \qquad C \text{ any real \#}$$

$$\ln|y-2| - \ln|y+2| = 4x + C$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + C$$

$$\left|\frac{y-2}{y+2}\right| = Ce^{4x} \qquad C > 0$$

$$\frac{y-2}{y+2} = Ce^{4x} \qquad C \neq 0$$

$$y-2 = (y+2)Ce^{4x}$$

$$y(1-Ce^{4x}) = 2Ce^{4x} + 2$$

$$y = \frac{2ce^{4x}+2}{1-ce^{4x}}$$

Looking back we see that

$y=2$ and $y=-2$ are both solutions

if we take $c=0$ then

$$y = \frac{2\cdot 0\cdot e^{4x}+2}{1-0\cdot e^{4x}} = 2.$$ $y=-2$ is not included in $y = \frac{2ce^{4x}+2}{1-ce^{4x}}$ and hence is called a "singular" solution

Interval of Validity

Let's take a moment to formally define solution to an n^{th}-order differential equation

DEF: Any function ϕ, defined on an interval I and possesing at least n derivatives that are continuous on I is a solution to an n^{th}-order DE $F(x, y, y', \dots, y^{(n)}) = 0$ if

$$F(x, \phi(x), \phi'(x), \dots, \phi^{(n)}(x)) = 0 \quad \text{for all } x \in I.$$

So, to each solution we attach an interval which we call the interval of validity (or definition or existence) We take this to be the largest interval on which $\phi(x)$ and its first n derivatives are continuous.

In the case of

$$y = \frac{2ce^{4x} + 2}{1 - ce^{4x}}$$

y is discontinuous when

$$1 - ce^{4x} = 0 \rightarrow e^{4x} = \frac{1}{c} \rightarrow x = -\frac{1}{4}\ln(c)$$

when $c > 0$

There is no discontinuity when $c \leq 0$ since in this case $e^{4x} = \frac{1}{c}$ has no solution.

When $c > 0$ we need to take

$(-\infty, -\frac{1}{4}\ln(c))$ or $(-\frac{1}{4}\ln(c), \infty)$ as the interval of validity.

For initial value problems we take the interval of validity to be the one which contains our initial value

Soln (2):

$$(e^{2y} - y)\cos(x)\frac{dy}{dx} = e^y \sin(2x), \quad y(0) = 0$$

$$\frac{e^{2y} - y}{e^y}\frac{dy}{dx} = \frac{\sin(2x)}{\cos(x)}$$

$$\int \frac{e^{2y} - y}{e^y}\,dy = \int e^y - ye^{-y}\,dy$$

integration by parts

$$\int ye^{-y}\,dy$$

$$= -ye^{-y} + \int e^{-y}\,dy$$

$$= -ye^{-y} - e^{-y}$$

$$= e^y + ye^{-y} + e^{-y}$$

double angle id

$$\int \frac{\sin(2x)}{\cos(x)}\,dx = \int \frac{2\sin(x)\cos(x)}{\cos(x)}\,dx$$

$$= \int 2\sin(x)\,dx = -2\cos(x)$$

$$e^y + ye^{-y} + e^{-y} = -2\cos(x) + C$$

Cannot solve for y! we leave it as an implicit solution.

$$y(0)=0$$

$$e^{y(0)} + y(0)e^{-y(0)} + e^{-y(0)} = -2\cos(0) + C$$

$$1+1 = -2 + C \rightarrow \quad C = 4$$

$$e^{y} + ye^{-y} + e^{-y} = -2\cos(x) + 4$$

2.3 First order linear equations

Paul's link: First Order Linear equations

A large majority of this course will be devoted to higher order linear equations. Their method of solution relies on the solution to first order linear equation.

$$y' + P(x)\, y = g(x)$$

We will see in class that the integrating factor $e^{\int P}$ is a key piece to solving first order linear equations that are in the above general form.

Note that if the equation is in any other form the integrating factor will not work.

The point of the integrating factor is that when we multiply the entire equation by it we can apply the product rule on the left hand side.

$$e^{\int P} y' + P e^{\int P} y = \frac{d}{dx}\left(e^{\int P} y\right)$$

Note that I have employed suppression of variables. That is, I am writing P instead of $P(x)$. It should be understood that y, g, P are all functions in x.

Thus our equations becomes

$$\frac{d}{dx}\left(e^{\int P} y\right) = g e^{\int P}.$$

We then integrate both sides with respect to x. On the left hand side, this simply undoes the differentiation.

$$e^{\int P} y = \int g e^{\int P} dx + C$$

We then multiply both sides by $e^{-\int P}$ to get an *explicit* solution. Thus if a linear equation is solvable, it has an explicit solution.

$$y = e^{-\int P} \int g e^{\int P} dx + C e^{-\int P}$$

WHEN SOLVING FIRST ORDER DIFFERENTIAL EQUATIONS DO NOT SIMPLY APPLY THIS FORMULA. WORK IT OUT FROM THE BEGINNING EVERY TIME LIKE I SHOWED YOU IN CLASS.

Def: (linear 1st order equation)

A first order differential equation of the form

$$\frac{dy}{dx} + P(x)y = g(x)$$

is said to be a linear equation in the variable y.

→ The left hand side nearly looks like the product rule!

$$\frac{d}{dx}\left(y\mu(x)\right) = y'\mu(x) + \mu'(x)y = y'\mu(x) + P(x)\mu(x)y$$

We want these to be the same

$$\mu' = P(x)\mu$$

This is a separable equation, which we know how to solve!

$$\frac{1}{\mu}\mu' = P(x) \rightarrow \int \frac{1}{\mu}\,d\mu = \int P(x)\,dx$$

$$\rightarrow \ln|\mu| = \int P(x)\,dx + c$$

$$\rightarrow \mu(x) = ce^{\int P(x)\,dx}$$

Def: (integrating factor)

For a general 1st-order linear equation

$$\frac{dy}{dx} + P(x)y = g(x),$$

the function $\mu(x) = e^{\int P(x)\,dx}$ is called the integrating factor.

The general method of solution for 1st order linear equation, $\frac{dy}{dx} + P(x)y = g(x)$, is as follows

$$\frac{dy}{dx} + P(x)y = g(x)$$

multiply by int factor

$$e^{\int P}\frac{dy}{dx} + e^{\int P}P(x)y = e^{\int P}g(x)$$

apply product rule (backwards)

$$\frac{d}{dx}\left(e^{\int P}y\right) = e^{\int P}g(x)$$

integrate both sides

$$\int \frac{d}{dx}\left(e^{\int P}y\right)dx = \int\left[e^{\int P}g(x)\right]dx$$

$$e^{\int P}y = \int\left[e^{\int P}g(x)\right]dx + C$$

divide by $e^{\int P(x)dx}$

$$y = e^{-\int P(x)dx}\int\left[e^{\int P(x)dx}g(x)\right]dx + Ce^{-\int P(x)dx}$$

DO NOT MEMORIZE THIS FORMULA!

WORK IT OUT EVERYTIME!

Ex: Solve $\frac{dy}{dx} - 3y = 6$.

Soln: The equation is already in standard form.

$$\text{Int Fac} = e^{\int P(x)dx}$$

$$P(x) = -3 \rightarrow e^{\int -3dx} = e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - e^{-3x}3y = 6e^{-3x}$$

$$\frac{d}{dx}\left(e^{-3x}y\right) = 6e^{-3x}$$

$$e^{-3x}y = 6\int e^{-3x}\,dx + C$$

$$= -\frac{6}{3}e^{-3x} + C$$

$$y = -2 + ce^{3x}$$

$\int e^{-3x}\,dx = \int -\frac{1}{3}e^{u}\,du$

$u = -3x \qquad = -\frac{1}{3}e^{u}$

$du = -3\,dx \qquad = -\frac{1}{3}e^{-3x}$

Let's check that we did our work correctly.

$$y' = 3ce^{3x}, \quad y = -2 + ce^{3x}$$

so $y' - 3y = 6$ becomes

$$3ce^{3x} - 3(-2 + ce^{3x})$$

$$= 3ce^{3x} + 6 - 3ce^{3x}$$

$$= 6 \quad \checkmark$$

Ex: Solve $x\frac{dy}{dx} - 4y = x^6 e^x$.

Soln: This is not yet in standard form

We need it to look like $\frac{dy}{dx} - P(x)y = g(x)$

So we divide both sides by x

$$\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x \rightarrow P(x) = -\frac{4}{x}$$

We need to pick an interval on which $P(x)$ is continuous on which we will solve for y. Take $(0, \infty)$.

Now our integrating factor is

$$e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|} = x^{-4}$$

Our equation becomes

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x$$

$$\frac{d}{dx}\left(x^{-4}y\right) = xe^x$$

$$x^{-4}y = \int xe^x dx + C \quad \text{parts}$$

$$= xe^x - e^x + C$$

$$y = x^5 e^x - x^4 e^x + cx^4$$

What is the interval of validity?

Remember our restriction from $P(x)$ is $(0,\infty)$. y is differentiable on all of $(0,\infty)$ so no other restrictions need to be made.

Ex: Find the general solution to

$$(x^2-9)\frac{dy}{dx} + xy = 0$$

Soln: $\frac{dy}{dx} + \frac{x}{x^2-9}y = 0$

$p(x)$ is continuous on $(-\infty,-3)$, $(-3,3)$, $(3,\infty)$

ultimately our solution will work for $(-\infty,-3)$ and $(3,\infty)$.

int factor: $e^{\int \frac{x}{x^2-9}dx}$

$= e^{\frac{1}{2}\ln|x^2-9|}$

$= \sqrt{x^2-9}$ ← on $(-\infty,-3)\cup(3,\infty)$ $x^2-9>0$

$\int \frac{x}{x^2-9}dx = \int \frac{1}{2u}du$

$u = x^2-9$ $= \frac{1}{2}\ln|u|$

$du = 2x\,dx$ $= \frac{1}{2}\ln|x^2-9|$

$$\frac{d}{dx}\left(\sqrt{x^2-9}\,y\right) = 0$$

$$\sqrt{x^2-9}\,y = c$$

$$y = \frac{c}{\sqrt{x^2-9}}$$

This works as a general solution for either $(-\infty,-3)$ or $(3,\infty)$.

→ In-Class exercises

Example 2 & 3 from Paul's online math notes

2.5 Exact equations

Paul's link: Exact equations

We need to review some multivariable calculus.

Definition 2.5.1 (Partial derivatives)**.** *Let $f(x,y)$ be a function from $\mathbb{R}^2 \to \mathbb{R}$. (This is called a scalar-valued function because it's range is a set of scalars.) It's* partial derivative w.r.t. *x is the limit*

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

Likewise, it's partial derivative w.r.t. *y is the limit*

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Theorem 2.5.2 (*Clairaut's Theorem*)

Let $f(x,y)$ be a function from $\mathbb{R}^2 \to \mathbb{R}$. If f has continuous second partials then

$$\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Recall that

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

serves as a general form for an arbitrary first order linear equation.

Definition 2.5.3 (Exact equation)**.** *A first order differential equation*

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if there is a scalar function $f(x,y)$ such that

$$M(x,y) = \frac{\partial f}{\partial x} \quad \text{and}$$
$$N(x,y) = \frac{\partial f}{\partial y}$$

You might remember from calculus class that this makes $(M(x,y), N(x,y))$ a conservative vector field!

In order to check that an equation is exact, it is enough, by Theorem 2.5.2, to verify that $M_y = N_x$.

By the chain rule, the left hand side of the exact equation becomes

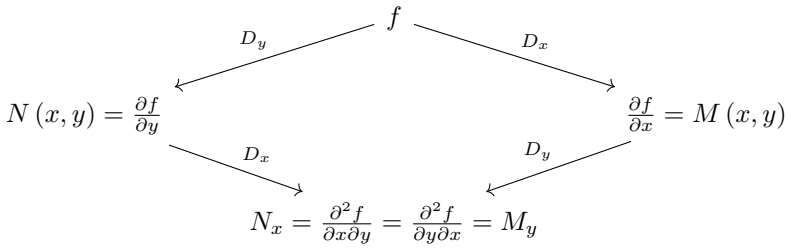
$$\frac{d}{dx} f(x,y) = 0.$$

After integrating both sides we see that

$$f(x,y) = c$$

is an implicit solution to the differential equation for all real numbers c.

There are many ways that f can be determined. It can be helpful to reference the diagram below.



My preferred method is to note that

$$f(x,y) = \int N dy + g(x) = \int M dx + h(y)$$

for some "constants" (they are constant w.r.t. the variable of integration) g and h. One can then use the second equality to determine exactly what g and h are.

Exact Equations

A class of first order (possibly non-linear) equations.

DEF: (The partial derivative)

Let $f(x,y)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}$

Then its partial derivatives are

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \quad \text{and}$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Basically, to calculate $\frac{\partial f}{\partial x}$ you treat y as a constant and then differentiate w.r.t x. For the other partial derivative $\frac{\partial f}{\partial y}$ you treat x as constant and differentiate w.r.t y.

Ex: $f(x,y) = x^3 \sin(x-y)$

$$\frac{\partial}{\partial x} f(x,y) = 3x^2 \sin(x-y) + x^3 \cos(x-y)(1-y)$$

$$\frac{\partial}{\partial y} f(x,y) = x^3 \cos(x-y)(x-1)$$

DEF: A first order differential equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is an exact equation if there is a function $f(x,y)$ such that

$$M(x,y) = \frac{\partial f}{\partial x} \quad \text{and}$$

$$N(x,y) = \frac{\partial f}{\partial y}$$

THEOREM

$M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

ID QUIZ

Which of the following are exact equations.

(a) $(2xe^{2y} - x\cos(xy) + 2y)dx + (e^{2y} - y\cos(xy))dy = 0$

(b) $\cos(x)y\,dx + \sin(x)\,dy = 0$ ← Also separable! Is it linear?

(c) $(x^2y^2 + x)\,dx + \left(\frac{2}{3}x^3y + y^{100}\right)dy = 0$

(d) $(y^2 - 6xy + \ln|x|)\,dx + (2y - 6x)dy = 0$

Note that if $f(x,y)$ is such that

$$\frac{\partial f}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x,y)$$

multi dim chain rule!

then $$\frac{d}{dx} f(x,y(x)) = \frac{\partial f}{\partial x}(x,y(x)) + \frac{\partial f}{\partial y}(x,y(x))\frac{dy}{dx}$$

$$= M(x,y(x)) + N(x,y(x))\frac{dy}{dx}$$

$$= M(x,y)\,dx + N(x,y)\,dy = 0$$

Thus $f(x,y) = c$ is an implicit solution to the exact solution.

Our goal then is to find f.

General method of soln:

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad (*)$$

and

$$\frac{\partial f}{\partial x} = M(x,y) \qquad (**)$$

1) Antidiff $(**)$ w.r.t x

$$f(x,y) = \int M(x,y)\,dx + g(y) \qquad (***)$$

where $g(y)$ is the constant of integration! Remember when we partially diff w.r.t x we treat ys as constants so, going backwards, our constant of integration must be dependent on y.

2) partially diff w.r.t. y

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + \frac{dg}{dy} = N(x,y)$$

Hence $\frac{dg}{dy} = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$

$$g(y) = \int \left[N(x,y) - \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right)\right] dy$$

Substituting in (**) we get

$$f(x,y) = \int M(x,y)\,dx + \int \left[N(x,y) - \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right)\right] dy$$

As always I do NOT recommend memorizing this formula.

EX: Solve $2xy\,dx + (x^2-1)\,dy = 0$

Soln:

$$\frac{\partial}{\partial y} 2xy = 2x$$

$$\| \checkmark$$

$$\frac{\partial}{\partial x}(x^2-1) = 2x$$

We have an exact equation!

We know there is an $f(x,y)$ s.t. $\frac{\partial f}{\partial x} = 2xy$

Appendix

DIFFERENTIATION RULES

General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}[cf(x)] = cf'(x)$
3. $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
4. $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$
5. $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$ (Product Rule)
6. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ (Quotient Rule)
7. $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ (Chain Rule)
8. $\frac{d}{dx}(x^n) = nx^{n-1}$ (Power Rule)

Exponential and Logarithmic Functions

9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = a^x \ln a$
11. $\frac{d}{dx}\ln|x| = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Trigonometric Functions

13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\tan x) = \sec^2 x$
16. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
17. $\frac{d}{dx}(\sec x) = \sec x \tan x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$

Inverse Trigonometric Functions

19. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}}$
20. $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}}$
21. $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$
22. $\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$
23. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$
24. $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$