Problem: Suppose the probability space (X, m) is nonatomic. Show that there are sets with arbitrarily small positive measure, that is, for every $\epsilon > 0$ there is a set $A \subset X$ with $0 < m(A) < \epsilon$.

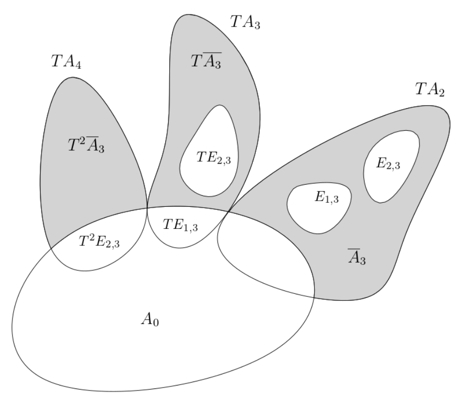
Solution based on notes from the course on ergodic theory I took with Máté Wierdl: Suppose, for contradiction, that there is a set, A, so that $\alpha := \inf\{m(B) : B \subset A \text{ and } m(B) \neq 0\} > 0$. We will use Ramsey's theorem from graph theory to construct a $B \subset A$ with $m(B) = \alpha$. Let $(B_n)_n \subset A$ be such that $\lim_n m(B_n) = \alpha$. We may choose this sequence so that $m(B_n) < 2\alpha$ for all n. Consider the graph with vertices (B_n) with an edge connecting two vertices, B_m and B_n, if and only if $m(B_m \cap B_n) > 0$. By our assumption, the measure of their intersection must be at least α. By Ramsey's theorem there is an infinite independent set or an infinite clique. In a probability space, a collection of disjoint sets must be finite. So there is a subsequence, $(B_{n_k})_k$ where no two of the B_{n_k}'s are disjoint. That is $m\left(B_{n_k} \cap B_{n_j}\right) \geq \alpha$ for all k, j. In fact, any finite intersection of the B'_{n_k}s must have measure at least α. Indeed, suppose $m\left(\cap_k^K B_{n_k}\right) \geq \alpha$. Then, $m\left(\cap_{k=1}^{K+1} B_{n_k}\right) = 0$ implies $m\left(\cap_{k=1}^{K+1} B_{n_k} \cup \left[B_{n_{K+1}} \cap B_1\right]\right) \geq 2\alpha$. But this union is a subset of B_1 which has measure strictly less than 2α. Contradiction. By induction $m\left(\cap_k^K B_{n_k}\right) \geq \alpha$ for all K. Then we have $\alpha \leq \lim_K m\left(\cap^K B_{n_k}\right) = m\left(\cap_k B_{n_k}\right) \leq \lim_k m\left(B_{n_k}\right) = \alpha$. Thus $\cap B_{n_k}$ is atomic.

THM: Let (X, m, T) be an ergodic, non-atomic system. Assume T is invertible and that T^{-1} is also measure preserving. Given $\epsilon > 0$, $H \in \mathbb{N}$ there is a $B \subset X$ such that $B, TB, \ldots, T^{H-1}B$ are pairwise disjoint and $m\left(\cup_{n<H} T^n B\right) > 1 - \epsilon$.

Proof: Let $A_0 \subset X$ be such that $0 < m(A_0) < \frac{\epsilon}{H-1}$. Define $A_{n+1} = TA_n \setminus (\cup_{i=0}^n A_i)$ for $n \geq 0$. Note that by this construction $A_0, A_1, \ldots$ are pairwise disjoint. As T is a measure preserving transformation, we have that $TA_0, TA_1, \ldots$ are also pairwise disjoint. Suppose not. Then there are n, m so that $m(TA_m \cap TA_n) > 0$. But then we would have $m(A_m \cap A_n) > 0$. Contradiction. From this we have that $A_{n+1} = TA_n \setminus A_0$. Observe, $TA_n \cap A_i = TA_n \cap \left(TA_{i-1} \setminus \cup_j^n A_j\right) = \varnothing$ for all $0 < i \leq n$.

Define $E_{k,n} = T^{-k}\left(T\left(A_{n+k-1}\right) \cap A_0\right)$ for all n and $k \in [1, H-1]$. Define $\overline{A}_n = A_n \setminus \cup_{k=1}^{H-1} E_{k,n}$. Define out base set as

$$B = \cup_{n \equiv_H 0} \overline{A}_n.$$



An example of what the $\overline{A}_n$ would look like in the case when $n = 3$ and $H = 3$ is given to the right. Notice that each of our error sets as well as the first $H-2$ forward images will be removed among the A_n to get the $\overline{A}_n$ but that only the removal of the $E_{k,Hn}$ is occurring in B. In the specific case of our illustration $TE_{1,3} = T^2E_{2,2} = TA_3 \cap A_0$. However in B only $E_{1,3}$ will be removed.

We have four things to prove to get our inequality:

1. $m\left(\cup_n A_n\right) = 1$
2. $m\left(\cup_{k=1}^{H-1} \cup_n E_{k,n}\right) < \epsilon$
3. $\left(\cup_n A_n\right) \setminus \left(\cup_m \cup_{k=1}^{H-1} E_{k,m}\right) = \cup_n \overline{A}_n$
4. $m\left(\cup_n \overline{A}_n\right) \leq m\left(\cup_{n=0}^{H-1} T^n B\right)$

By the injectivity of T, we have $T(A \setminus B) = TA \setminus TB$. This allows us the following,

$$A_i = TA_{i-1} \setminus A_0 \subset TA_{i-1} = T^2 A_{i-2} \setminus TA_0 \subset T^2 A_{i-2} = \cdots \subset T^i A_0 \subset \cup_{n=0} T^n A_0$$

for all $i > 0$. Clearly, $A_0 \subset \cup_{n=0} T^n A_0$. Therefore, $\cup_n A_n \subset \cup_n T^n A_0$. We go by induction for the converse. The basecase is $A_0 \subset \cup_{n=0} T^n A_0$. Suppose $T^k A_0 \subset \cup_{n=0} A_n$. Then

$$T^{k+1} A_0 = T\left(T^k A_0\right) \subset T \cup_{n=0} A_n = \cup_{n=0} T A_n \subset \cup_{n=0} \left(A_{n+1} \cup A_0\right) = \cup_{n=0} A_n.$$

Thus by induction, $T^k A_0 \subset \cup_n A_n$ for all k. So we have proved one. Consequently, as $\cup_n T^n A_0$ is invariant and has positive measure, $m\left(\cup_n A_n\right) = m\left(\cup_n T^n A_0\right) = 1$.

We move on to two. By definition, $T^k E_{k,n} = T\left(A_{n+k-1} \cap A_0\right) \subset A_0$ for all n and $1 \le k \le H-1$. As the T^{-1} is measure preserving and since the $T^k E_{k,n}$ are disjoint in k, $m\left(\cup_{k=1}^{n} \cup_n E_{k,n}\right) = m\left(\cup_{k=1}^{H-1} \cup_n T^k E_{k,n}\right) = \sum_{k=1}^{H-1} m\left(\cup_n T^k E_{k,n}\right) \le (H-1)\, m\left(A_0\right) < \epsilon$.

For three, note that $E_{k,n} \cap A_m = \varnothing$ for all $m \ne n$ and $1 \le k \le H-1$. Thus we have,

$$\cup_n A_n \setminus \left(\cup_m \cup_{k=1}^{H-1} E_{k,m}\right) = \cup_n \left(A_n \setminus \left(\cup_m \cup_{k=1}^{H-1} E_{k,m}\right)\right) = \cup_n \left(A_n \setminus \cup_{k=1}^{H-1} E_{k,n}\right) = \cup_n \overline{A}_n.$$

The last thing we need to show is that $m\left(\cup \overline{A}_n\right) \le m\left(\cup_{n=0}^{H-1} T^n B\right)$. For this it is important to realize that $m\left(A_n\right) \ge m\left(A_{n+1}\right)$ and $T^{H-1} \overline{A}_n = A_{n+H-1}$. We then have

$$m\left(\cup_{n=0} \overline{A}_n\right) = \sum_{n=0} m\left(\overline{A}_n\right) = \sum_{n=H-1}^{\infty} m\left(A_n\right) \le H \sum_{n=0}^{\infty} m\left(A_{nH-1}\right) = Hm\left(B\right) = m\left(\cup^{H-1} T^n B\right).$$

Putting all of this together yields

$$1 - \epsilon < 1 - m\left(\cup_{k=1}^{H-1} \cup_m E_{k,m}\right) = m\left(\cup_n A_n \setminus \cup_{k=1}^{H-1} \cup_m E_{k,m}\right) = m\left(\cup \overline{A}_n\right) \le m\left(\cup_{n=0}^{H-1} T^n B\right).$$

Finally, if $n \ne m$ with $0 < n, m \le H-1$ then

$$T^n B \cap T^m B = \left(\cup_{k \equiv_H 0} T^n \overline{A}_k\right) \cap \left(\cup_{k \equiv_H 0} T^m \overline{A}_k\right) \subset \left(\cup_{k \equiv_H 0} A_{n+k}\right) \cap \left(\cup_{k \equiv_H 0} A_{m+k}\right) = \varnothing.$$

$\square$

Note: I initially tried to define $A_n = TA_{n-1} \setminus A_0$ and then show that the A_n were disjoint. However my attempt at proving the disjointness was messy and hard to read. Dr. Wierdl sent me an email with a solution that I think reads well.

Proof of disjointness from Mate: Let $A_n = TA_{n-1} \setminus A_0$. If we had $A_2 \cap A_1 \ne \emptyset$ this would mean that for some $x_1 \in A_1$ and $x_0 \in A_0$ we'd have $Tx_1 = Tx_0$ which implies $x_0 = x_1$. If we had $A_3 \cap A_1 \ne \emptyset$ then for some $x_2 \in A_2$ and $x_0 \in A_0$ we'd have $Tx_2 = Tx_0$ hence $x_2 = x_0$ but then $A_2 \cap A_0 \ne \emptyset$. If we had $A_3 \cap A_2 \ne \emptyset$ then for some $x_1 \in A_1$ and $x_2 \in A_2$, we'd have $Tx_1 = Tx_2$ so $x_1 = x_2$ so $A_1 \cap A_2 \ne \emptyset$ which we already know is not the case. Etc.

Note from Máté: If I now take an arbitrary $x_0 \in A_0$ and I find that $T^n x_0 \notin A_0$ for all n, then $x_0 \in T^{-n} A_n$ for all n, that is, $x_0 \in \cap_n T^{-n} A_n$. This implies that the measure of such x_0 is 0. So almost all points $x \in A_0$ return to A_0, that is, $T^n x \in A_0$. This we knew (by Poincaré recurrence) but I haven't seen it this way.

Note from me: This leads into approximation of dynamical systems. The last part of the text by Cornfeld, Fomin and Sinai has some interesting things about this.