The Wallman Compactification

Revised notes for the graduate student seminar

Conner Griffin

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These are revised and expanded notes for a talk I gave at the University of Memphis' Department of Math Sciences' graduate student seminar. The talk was on April 7th, 2023. These notes are meant to be a modern expression of the paper "Lattices and Topological Spaces" by Henry Wallman. The slant of these notes is toward the application of ultrafilters in combinatorics so that it may serve as a reference for some results and techniques that are useful in that area. Thus, some consequences of that paper dealing with homology are omitted.

Note 1. *Remember that a topological space, X, is*

- T_1 *(Frechet) if and only if for any $x, y \in X$ there is an open neighborhood U around x with $y \notin U$.*
- T_2 *(Hausdorff) if and only if for any $x, y \in X$ there is an open neighborhood, U, of x and an open neighborhood, V, of y with $U \cap V = \varnothing$.*
- T_4 *(normal) if and only if for any disjoint closed sets $E, F \subset X$ there is an open neighborhood, U, of E and an open neighborhood, V, of F with $U \cap V = \varnothing$.*

Definition 1 (Stone-Čech compactification)**.** *For a topological space, X, the* Stone-Čech compactification *of X, which we denote βX, is a compact Hausdorff space together with a continuous function, $e : X \to \beta X$, satisfying the property which follows. Let K be a compact Hausdorff space. For any continuous function, $f : X \to K$, there is a continuous function, $F : \beta X \to K$ with $F \circ e = f$ (or $F|_{e(X)} = f$). In other words the diagram below commutes.*

$$\begin{tikzcd} X \arrow[r, hook, "e"] \arrow[dr, "f"'] & \beta X \arrow[d, dashed, "F"] \\ & K \end{tikzcd}$$

Definition 2 (poset)**.** *A* partially ordered set *or* poset *is an ordered pair,* $(L, \preceq)$*, of a set,* L*, and a reflexive, antisymmetric, transitive relation,* $\preceq$*. A relation satisfying these axioms is called a partial ordering.*

Example 1. *Let* S *be a set. Then* $\mathcal{P}(S) := \{A : A \subseteq S\}$ *together with* $\subseteq$ *is a poset.*

Definition 3 (lattice)**.** *A poset,* $(L, \preceq)$*, is a* lattice *when for any* $a, b \in L$ *there is a* $j := a \vee b$ *such that* $a, b \preceq j$ *and if* $a, b \preceq k$ *for some* k *then* $j \preceq k$*. That is, for any* $a, b \in L$*, the set of all elements in* L *which are greater than both* a *and* b *has a minimal element. We call such a* j *the join of* a *and* b*. Likewise, for any* $a, b \in L$ *there is an* $m := a \wedge b$ *such that* $m \preceq a, b$ *and if* $l \preceq a, b$ *then* $l \preceq m$*. That is, for any* $a, b \in L$*, the set of all elements in* L *which are less than both* a *and* b *has a maximal element. We call such an* m *the meet of* a *and* b*.*

Example 2. *For a set,* S*,* $(\mathcal{P}(S), \subseteq)$ *is a lattice with* $A \cup B$ *being the join of* A *and* B *and* $A \cap B$ *being the meet of* A *and* B*.*

Example 3. *For a topological space,* X*,* $\kappa(X) := \{A \subset X : A \text{ is closed in } X\}$ *together with* $\subseteq$ *is a lattice with* $A \cup B$ *and* $A \cap B$ *again being the join and meet of* A *and* B*.*

Definition 4 (distributive lattice)**.** *A lattice,* L*, is distributive when*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

or equivalently

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all $x, y, z \in L$*.*

Example 4. *That* $(\mathcal{P}(S), \subseteq)$ *and* $(\kappa(S), \subseteq)$ *are distributive is well known:*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

for all $A, B, C \in \mathcal{P}(S)$

Definition 5 (filter)**.** *A* filter*,* $\mathcal{F}$*, of a lattice,* $(L, \preceq)$*, is a nonempty subset of* L *satisfying:*

1. $\mathcal{F}$ *is downward directed: If* $x, y \in \mathcal{F}$ *then* $x \wedge y \in \mathcal{F}$*.*

2. $\mathcal{F}$ *is upward closed: Let* $x \in \mathcal{F}$ *and* $y \in L$*. If* $x \preceq y$ *then* $y \in \mathcal{F}$*.*

A filter is called proper *if it is a proper subset of the lattice.*

Note 2. *Let L be a lattice. If a is a minimal element of L then a is the unique minimum element of L. Indeed, suppose $a \in L$ is minimal. For all $b \in L$, $a \wedge b \preceq a$ and thus $a \wedge b = a$. This is equivalent to saying $a \preceq b$ for all $b \in L$.*

Definition 6 (0)**.** *If it exists, we call the minimum element of a lattice* 0.

Definition 7 (finite intersection property)**.** *Let L be a lattice with* 0. *We say $\mathcal{F} \subset L$ has the finite intersection property if and only if $a \wedge b \neq 0$ for all $a, b \in \mathcal{F}$.*

Note 3. *A filter on a lattice with a minimum element,* 0, *is proper if and only if it does not contain* 0. *In the case of $\mathcal{P}(S)$ and $\kappa(S)$, $0 = \varnothing$.*

Definition 8 (closed filter)**.** *A* filter *on a set, S, is a proper filter of the lattice $(\mathcal{P}(S), \subseteq)$. A* closed filter *on a topological space, X, is a proper filter of the lattice $(\kappa(X), \subseteq)$.*

Definition 9 (ultrafilter)**.** *An* ultrafilter *on a lattice is a proper filter which is maximal among all proper filters. A* closed ultrafilter *is an ultrafilter on $(\kappa(X), \subseteq)$.*

Theorem 1. *Let L be a lattice with* 0. *Let $\mathcal{C} \subsetneq L$ be such that for all $a, b \in \mathcal{C}$, $a \wedge b \in \mathcal{C}$. If $c \in L \setminus \mathcal{C}$ is such that $a \wedge c \neq 0$ for all $a \in \mathcal{C}$, then*

$$u(\mathcal{C}, c) := \{x : a \wedge c \preceq x \text{ for some } a \in \mathcal{C}\}$$

is a proper filter that properly contains $\mathcal{C}$.

Proof. Let L be a lattice with 0. Let $\mathcal{C} \subsetneq L$ be such that for all $a, b \in \mathcal{C}$, $a \wedge b \in \mathcal{C}$. Suppose $c \in L \setminus \mathcal{C}$ is such that $a \wedge c \neq 0$ for all $a \in \mathcal{C}$. Let $\mathcal{F} = \{x : a \wedge c \preceq x \text{ for some } a \in \mathcal{C}\}$. If $x, y \in \mathcal{F}$ then there are $a_x,\ a_y \in \mathcal{C}$ such that $a_x \wedge c \preceq x$ and $a_y \wedge c \preceq y$. Then $a_x \wedge a_y \wedge c \preceq x \wedge y$. Since $a_x \wedge a_y \in \mathcal{C}$, $x \wedge y \in \mathcal{C}$. Let $x \in \mathcal{F}$ and $y \in L$ with $x \preceq y$. Then there is a $a_x \in \mathcal{C}$ with $a_x \wedge c \preceq x \preceq y$. So, $y \in \mathcal{F}$. If $a \in \mathcal{C}$, then $a \wedge c \preceq a$. As well, $a \wedge c \preceq c$ for all $a \in \mathcal{C}$. Therefore, by the previous two lines, $\mathcal{C} \subsetneq \mathcal{F}$. Finally, $a \wedge c \neq 0$ for all $a \in \mathcal{C}$ and so $0 \notin \mathcal{F}$. □

Theorem 2. *Let L be a lattice with* 0. *The following are equivalent:*

1. *p is an ultrafilter on L*
2. *p has the finite intersection property and for any $a \in L \setminus p$ there is a $b \in p$ with $a \wedge b = 0$.*
3. *p is maximal with respect to the finite intersection property.*

Proof. $1 \Rightarrow 2$

Suppose p is an ultrafilter on L. Suppose for contradiction that there is an $a \in L \setminus p$ with $a \wedge b \neq 0$ for all $b \in p$. By Theorem 1, p cannot be an ultrafilter. Contradiction.

$2 \Rightarrow 3$

Suppose p has the finite intersection property and for any $a \in L \setminus p$ there is a $b \in p$ with $a \wedge b = 0$. Suppose for contradiction that there is a $\mathcal{C} \subset L$ with the finite intersection property so that $p \subsetneq \mathcal{C}$. Take $a \in \mathcal{C} \setminus p$. Then $a \wedge b \neq 0$ for all $b \in p$. Contradiction.

$3 \Rightarrow 1$

Suppose p is maximal with respect to the finite intersection property. If p is a filter then it must be an ultrafilter, since the family of all filters is a subfamily of the family of all collections with the finite intersection property. For contradiction suppose p is not a filter. Then it is either not downward directed or not upward closed. Suppose the former. Then there is an $x, y \in p$ so that $x \wedge y \notin p$. Suppose there is a $b \in p$ so that $x \wedge y \wedge b = 0$. Then p does not have the finite intersection property. Thus $p \cup \{x \wedge y\}$ has the finite intersection property contradicting the maximality of p. On the other hand, if p is not upward closed then there is an $x \in p$ and a $x \preceq y$ such that $y \notin p$. But, because $x \wedge z \neq 0$ for all $z \in p$ and $x \wedge z \preceq y \wedge z$ it must be that $p \cup \{z\}$ has the finite intersection property. Contradiction. It has to be that p is a filter and hence an ultrafilter. □

Theorem 3. *Let L be a lattice with* 0*. Let $\mathcal{F} \subseteq L$ have the finite intersection property. Then there is an ultrafilter p so that $\mathcal{F} \subseteq p$.*

Proof. (By Zorn's Lemma) Let $\Gamma = \{\mathcal{G} \subseteq L : \ \mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{G}$ has the finite intersection property$\}$. Then $\mathcal{F} \in \Gamma$. Let $\mathcal{A}$ be a totally ordered subset of Γ. Then $\mathcal{F} \subseteq \bigcup_{\mathcal{G} \in \mathcal{A}} \mathcal{G}$. As well, take $a, b \in \bigcup_{\mathcal{G} \in \mathcal{A}} \mathcal{G}$. Since $\mathcal{A}$ is totally ordered there has to be a $\mathcal{G} \in \mathcal{A}$ so that $a, b \in \mathcal{G}$. So, $a \wedge b \neq 0$. The conditions of Zorn's Lemma are met and we now have a maximal p in Γ. Clearly, p is maximal with respect to the finite intersection property and hence is an ultrafilter. □

Definition 10 (prime)**.** *Let L be a lattice. Let $\mathcal{C} \in \mathcal{P}(L)$. We call $\mathcal{C}$* prime *if and only if $x \vee y \in \mathcal{C}$ implies $x \in \mathcal{C}$ or $y \in \mathcal{C}$.*

For the subset lattice, this quality is often called "partition regular." The following popular definition of partition regularity is slightly weaker than primality.

Definition 11 (partition regular)**.** *Let S be a set. Let $\mathcal{C} \subseteq \mathcal{P}(S)$ and let $\mathcal{C}^{\uparrow} := \{A :$ there is a $B \in \mathcal{C}$ such that $B \subseteq A\}$. Then $\mathcal{C}$ is* partition regular *if and only if $A \cup B \in \mathcal{C}$ implies $A \in \mathcal{C}^{\uparrow}$ or $B \in \mathcal{C}^{\uparrow}$.*

Theorem 4 (The relationship between maximality and primality)**.** *Let L be a distributive lattice with* 0.

1. *Ultrafilters on L are prime.*

2. *Let S be a set. Prime filters on S are ultrafilters on S.*

Proof of 1. Let p be an ultrafilter on a distributive lattice with 0. Suppose for contradiction that there is an x and a y in the lattice such that $x \vee y \in p$ but $x \notin p$ and $y \notin p$. Suppose that $a \wedge x \neq 0$ for all $a \in p$. Then $\{\alpha : \ a \wedge x \preceq \alpha \text{ for some } a \in p\}$ is a proper filter that properly contains p, contradicting the maximality of p. Therefore, there must be an $a_x \in p$ with $a_x \wedge x = 0$. Likewise, there is an $a_y \in p$ so that $a_y \wedge y = 0$. By the filter axioms, $(a_x \wedge a_y) \wedge (x \vee y) \in p$. However, because our lattice is distributive,

$$\begin{aligned}(a_x \wedge a_y) \wedge (x \vee y) &= (a_x \wedge a_y \wedge x) \vee (a_x \wedge a_y \wedge y) \\ &= (0 \wedge a_y) \vee (0 \wedge a_x) \\ &= 0 \vee 0 = 0 \in p.\end{aligned}$$

Thus $p = L$. This contradicts that ultrafilters are proper filters. □

Proof of 2. Let S be a set. Let $\mathcal{F}$ be a prime filter on S. Suppose for contradiction that $\mathcal{F}$ is not an ultrafilter. Then there is an ultrafilter, p with $\mathcal{F} \subsetneq p$. In other words, $p \setminus \mathcal{F} \neq \varnothing$. Take $A \in p \setminus \mathcal{F}$. Then $A \notin \mathcal{F}$. By the primality of $\mathcal{F}$, $S = A \cup (S \setminus A) \in \mathcal{F}$ and $A \notin \mathcal{F}$ implies that $S \setminus A \in \mathcal{F}$. But then A and $(S \setminus A)$ are both in p. Hence, $\varnothing = A \cap (S \setminus A) \in p$. Contradiction. □

Conjecture 1. *Let X be a topological space. Prime closed filters on X need not be closed ultrafilters.*

Definition 12 (Wallman compactification)**.** *Let X be a T_1 space. Define $\omega X := \{p \subset \kappa(X) : p \text{ is a closed ultrafilter}\}$. Take the topology on ωX to be the one with closed basis consisting of all sets of the form $\{p \in \omega X : A \in p, \ A \in \kappa(X)\}$. We call ωX together with this topology the* Wallman compactification *of X.*

Proof. We seek to prove that for X, a T_1 space, ωX is compact and that there is an embedding of X which is dense in ωX. Consider

$$\begin{aligned}h &: X \to \omega X \\ x &\mapsto \{A : \ x \in A, \ A \text{ is closed}\}.\end{aligned}$$

For any $x \in X$, $p_x := \{A : \ x \in A, \ A \text{ is closed}\}$ is a closed ultrafilter. Suppose $A \in p_x$, $A \subset B$ and B is closed. Then $x \in B$ so $B \in p_x$. That is p_x is upward closed. If $A, B \in p_x$ then $x \in A \cap B$ and $A \cap B$ is closed.

So, $A \cap B \in p_x$. The image of x under h is therefore a proper filter. Suppose $p_x \subsetneq q \in \omega X$. Then there is a $B \in q \backslash p_x$. This B does note contain x, but X is T_1. A topological space is T_1 if and only if all of its singleton sets are closed. Hence, $\{x\} \in p_x$. This is a contradiction because we would then have $\varnothing = B \cap \{x\} \in q$. So, our function h is well defined. Also, it is clearly injective.

Let A be a closed subset of X. Then $h(A) = \bigcup_{x \in A} \{h(x)\} \subset \{p \in \omega X : A \in p\}$. Suppose there is a closed set $K \subset \omega X$ with $h(A) \subset K \subsetneq \{p \in \omega X : A \in p\}$. Since K is closed it must be an arbitrary intersection of sets from the closed base of our topology. Therefore, $K = \bigcap_{B \in \mathcal{C}} \{p \in \omega X : B \in p\} = \{p \in \omega X : \mathcal{C} \subseteq p\}$. Take $q \in \{p \in \omega X : A \in p\} \setminus K$. Then there must be a $B_q \in \mathcal{C}$ with $B_q \notin p$. By the upward closure property of ultrafitlers, $A \nsubseteq B_q$. Take $x \in A \setminus B_q$. Then $h(x) \in h(A) \subseteq K$ which implies that $B_q \in h(x)$. But then $x \in B_q$. This is a contradiction. It must be that $\{p \in \omega X : A \in p\}$ is the smallest closed set containing A. In other words, $\overline{h(A)} = \{p \in \omega X : A \in p\}$. Since $\overline{h(A)} \bigcap h(X) = h(A)$ form a basis for the subspace topology on $h(X)$, h is naturally continuous. As well, $\overline{h(X)} = \{p \in \omega X : X \in p\} = \omega X$.

We still need that ωX is compact. To that end, we will show that any collection of closed sets in ωX which has the finite intersection property has nonempty intersection. Actually, we will show, without loss of generality, that if $\mathcal{E} := \{\overline{h(A)} : A \in E \subset \kappa(X)\}$ has the finite intersection property then $\bigcap_{A \in E} \overline{h(A)} \neq \varnothing$. Note that

$$\bigcap_{A \in E} \overline{h(A)} = \{p \in \omega X : \bigcap_{A \in F} A \in p, \text{ for all } F \in \mathcal{P}_f(E)\}$$

where $\mathcal{P}_f(E) := \{F \subset E : \varnothing \neq F, |F| < \infty\}$. Indeed, if $p \in \bigcap_{A \in E} \overline{h(A)}$ then $A \in p$ for all $A \in E$ and thus $\bigcap_{A \in F} A \in p$ for any $F \in \mathcal{P}_f(E)$. Likewise, if $\bigcap_{A \in F} A \in p$ for all $F \in \mathcal{P}_f(E)$ then $A \in p$ for all $A \in E$. So, $p \in \bigcap_{A \in E} \overline{h(A)}$. So, we will show that $\{p \in \omega X : \bigcap_{A \in F} A \in p, \text{ for all } F \in \mathcal{P}_f(E)\} \neq \varnothing$. By definition of the finite interesection property for $\mathcal{E}$, for any $F \in \mathcal{P}_f(E)$,

$$\varnothing \neq \bigcap_{A \in F} \overline{h(A)} = \overline{h(\cap_{A \in F} A)} = \{p \in \omega X : \bigcap_{A \in F} A \in p\}.$$

So, E has the finite intersection property and, therefore, there is a closed ultrafilter q with $E \subseteq q$. Since q is closed under finite intersections $\mathrm{FI}(E) := \{\bigcap_{A \in F} A : F \in \mathcal{P}_f(E)\} \subset q$. Thus $q \in \{p \in \omega X : \bigcap_{A \in F} A \in p, \text{ for all } F \in \mathcal{P}_f(E)\}$.

For a T_1 space, the Wallman compactficiation is a compactification. $\square$

Definition 13 (Basic open sets of ωX)**.** *Let X be a T_1 space and ωX the Wallman compactification of X. For all open U in X define $U^\circ := \{p \in \omega X : \forall B \in p, B \cap U \neq \varnothing\}$.*

Theorem 5. *Let X be a T_1 space and ωX the Wallman compactification of X. For any $A \in \mathcal{K}(X)$, $\omega X \setminus \overline{A} = (X \setminus A)^{\circ}$. Here $\overline{A} = \overline{h(A)}$ for convenience and aesthetics.*

Proof. Suppose $p \in \omega X$ is such that for all $B \in p$, $B \cap (X \setminus A) \neq \varnothing$. Of course A does not meet $X \setminus A$ so $A \notin p$. Thus $p \in \omega X \setminus \overline{A}$.

Conversely, suppose $p \in \omega X$ such that $A \notin p$. There must then be a closed $B \in p$ such that $B \cap A = \varnothing$. Thus $B \subset X \setminus A$. Well, for every $C \in p$, $\varnothing \neq C \cap B \subset C \cap X \setminus A$. Thus $p \in (X \setminus A)^{\circ}$. □

Theorem 6. *For a topological space, X, ωX is homeomorphic to βX if and only if X is normal.*

Proof. Let $h : X \to \omega X$ be defined as above. Let K be any compact Hausdorff space and $f : X \to K$ a continuous function. Let $p \in \omega X$. Consider the closed filter in K, $\mathcal{F}_p := \{A \subset K : \ A \text{ is closed and } f^{-1}(A) \in p\}$. This is a prime filter. Suppose $A \cup B \in \mathcal{F}_p$. Then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \in p$. Ultrafilters are prime so $f^{-1}(A) \in p$ or $f^{-1}(B) \in p$. By the continuity of f, either A is closed or B is closed in accord with which of $f^{-1}(A)$ or $f^{-1}(B)$ is in p. Thus $A \in \mathcal{F}_p$ or $B \in \mathcal{F}_p$.

Claim 1. *Prime filters on compact topological spaces contain unique cluster points.*

Proof of claim. Let p be a prime filter on a compact topological space. Filters on topological spaces have the finite intersection property. Thus $\bigcap_{A \in p} A \neq \varnothing$. So there exists a cluster point. Suppose $x \neq y \in p$ are both cluster points. Let $A \in p$. Then $x, y \in A$ and $A \setminus \{x\} \cup \{x\} \in p$. But, because p is prime $A \setminus \{x\} \in p$, in which case x is not a cluster point, or $\{x\} \in A$ in which case y is not a cluster point. There is exactly one cluster point. □

Let c_p be the unique cluster point associated with $\mathcal{F}_p$. Then the function $F : \omega X \to K$ defined by $F(p) = c_p$ is such that $F \circ h = f$. Indeed, $F(h(x)) = \bigcap_{A \in h(x)} A = f(x)$. As well, F is continuous. Let $A \subset K$ be closed. We want to see that $F^{-1}(A)$ is also closed. Well $F^{-1}(A) = \{p \in \omega X : \ f^{-1}(A) \in p\}$, so suppose $f^{-1}(A) \in p$. Then $A \in \mathcal{F}_p$ and hence $c_p \in A$. That is $F(p) \in A$. We have $\{p \in \omega X : \ f^{-1}(A) \in p\} \subseteq F^{-1}(A)$ For the other direction, suppose $p \in F^{-1}(A)$. Then $c_p \in A$. By upward closure, $A \cup (V \setminus A) \in \mathcal{F}_p$. It cannot be that $V \setminus A \in \mathcal{F}_p$ because it does not contain the cluster point. Therefore, $A \in \mathcal{F}_p$ and thus $f^{-1}(A) \in p$. So, $F^{-1}(A) = \{p \in \omega X : \ f^{-1}(A) \in p\}$ which is closed in ωX.

We are left to show that ωX is Hausdorff if and only if X is normal. Suppose X is normal. Let $p, q \in \omega X$ be distinct. For any $A \in p \setminus q$ there is a $B \in q \setminus p$ with $A \cap B = \varnothing$. Since X is normal there are open and disjoint $U, V \subset X$ such that $A \subset U$ and $B \subset V$. Then $p \in U^{\circ}$ and $q \in V^{\circ}$. By de Morgan's law, $U^{\circ} \cap V^{\circ} = \omega X \setminus \left(\overline{X \setminus U} \cup \overline{X \setminus V}\right)$. By disjointness and de Morgan's law, $X \setminus U \cup X \setminus V = X$. By the previous line and primality of ultrafilters, $X \setminus U \notin p$ implies $X \setminus V \in p$. Thus, $\overline{X \setminus U} \cup \overline{X \setminus V} = \omega X$. Finally, $U^{\circ} \cap V^{\circ} = \omega X \setminus \left(\overline{X \setminus U} \cup \overline{X \setminus V}\right) = \varnothing$. We have shown that ωX is Hausdorff whenever X is normal.

Suppose ωX is Hausdorff. Compact Hausdorff spaces are normal. Take disjoint and closed $E, F \subset X$. Then $\overline{h(E)}$ and $\overline{h(F)}$ are disjoint closed subsets of ωX. Then there are disjoint and open $U, V \subset \omega X$ such that $\overline{h(E)} \subset U$ and $\overline{h(F)} \subset V$. Of course, $h^{-1}(U)$ and $h^{-1}(V)$ are open and disjoint and contain E and F, respectively. It must be that X is normal. $\square$