Ergodic Theorems for Amenable Group Actions

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Table of Contents

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Topological groups

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- A topological space, X, is locally compact if every point $x \in X$ has a neighborhood, $\mathcal{O}$, so that $x \in \mathcal{O} \subset K$ for some compact K.
- Let G be a locally compact topological group acting on a measure space $(X, \mathcal{B}, \mu)$. Then the action is said to be *ergodic* if for any measurable A with $\mu\left(g^{-1}A \triangle A\right) = 0$ for all $g \in G$ we have $\mu(A) = 0$ or 1. [2]

Definition

Let G be a locally compact group. A *left (respectively right) Haar measure* on G is a Borel measure, m, with the following properties:

1. $m(gB) = m(B)$ (respectively $m(Bg) = m(B)$) for all Borel sets, B, and all $g \in G$
2. $m(K) < \infty$ for all compact sets $K \subset G$.
3. $m(\mathcal{O}) > 0$ for all $\mathcal{O} \subseteq G$.

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- For a discrete group, Haar measure is simply counting measure.

Definition

A σ-locally compact group G is *amenable* if, for any compact subset $K \subset G$ and $\varepsilon > 0$, there is a measurable set $F \subset G$ with compact closure such that KF is a measurable set with

$$m\left(F \triangle KF\right) < \varepsilon m\left(F\right),$$

where m denotes the left Haar measure on G.

- A measurable $F \subset G$ is said to be $(K, \varepsilon)-$invariant if $m(F \triangle KF) < \varepsilon m(F)$.

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- A locally compact group, G, is amenable if and only if every continuous G-action $G \to \text{Homeo}(X, d)$ on a compact metric space has an invariant probability measure. (the forward direction is a generalization of Krylov-Bogolyubov) [2]

examples of amenable groups

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- $SL\,(n, \mathbb{Z})$ is not amenable for all $n \geq 2$
- $\{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R}\} \subset SL(3, \mathbb{R})$ is an amenable group

[2, 4]

For the last example, $\{\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} : x, y, z \in \mathbb{R}\} \subset SL(3, \mathbb{R})$ is a solvable group and hence amenable.

We verify this by checking our commutator subgroups. Note that

$$\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -a & ab-c\\ 0 & 1 & -b\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -x & xy-z\\ 0 & 1 & -y\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & ay-bx\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

So our first commutator subgroup is generated by matrices of the form $\begin{pmatrix} 1 & 0 & ay - bx \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where $a, b, x, y \in \mathbb{R}$. Then

$$\begin{pmatrix} 1 & 0 & ay - bx \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \alpha\gamma - \beta\xi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & bx - ay \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \beta\xi - \alpha\gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So our second commutator subgroup is just the identity matrix and thus our group is solvable.

Let G be a σ-locally compact amenable group, with left Haar measure m, acting continuously on X, and let μ be a G-invariant probability measure on X. Define the unitary map $U_g : L^2_\mu \to L^2_\mu$ by $U_g\left(f\right)\left(x\right) = f\left(g^{-1}x\right)$. Let $\mathcal{I} = \cap_{g\in G} \ker U_{g^{-1}} - I$. Let $f_{\mathcal{I}}$ be the orthogonal projection of f onto $\mathcal{I}$.

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Then for any Følner sequence $\left(F_n\right)_n$ and $f \in L^2_\mu\left(X\right)$,

$$\frac{1}{m\left(F_n\right)} \int_{F_n} U_{g^{-1}} f dm\left(g\right) \to f_{\mathcal{I}}$$

in L^2_μ norm.

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Then for any Følner sequence $(F_n)_n$ and $f \in L^2_\mu(X)$,

$$\frac{1}{m(F_n)} \int_{F_n} U_{g^{-1}} f \, dm(g) \to f_{\mathcal{I}}$$

in L^2_μ norm.

Additionally, the action is ergodic if and only if

$$f_{\mathcal{I}} = \int_X f \, d\mu$$

for all $f \in L^2_\mu$.

proof idea:

As in the case of $\mathbb{Z}$ actions, U_g is unitary for each $g \in G$. Our space of invariant functions is orthogonal to $V = \cup_g \operatorname{Im} U_{g^{-1}} - I$. In fact, $L^2_\mu = I \oplus V$ and thus $f = f_{\mathcal{I}} + f_V$. Then we have that the ergodic averages of f_V converge to 0 as a consequence of the (K, ε)-invariance of the Følner sets and the ergodic averages of $f_{\mathcal{I}}$ converge to itself as it is invariant under every group action of G.

We can use results from Sana's presentation to see that we need some extra conditions on the Føner sequence in a pointwise theorem. Indeed, a consequence of the ultimate result that she discussed is that given an increasing sequence $(b_n) \subset \mathbb{N}$ and an ergodic transformation $T : [0,1] \to [0,1]$ there is a G_δ subset, $\mathcal{R}$, such that for all $A \in \mathcal{R}$

$$\limsup_n \frac{1}{b_n} \sum_{k=n+1}^{n+b_n} \mathbb{1}_A \left(T^k x\right) = 1 \ a.e.$$

and

$$\liminf_n \frac{1}{b_n} \sum_{k=n+1}^{n+b_n} \mathbb{1}_A \left(T^k x\right) = 0 \ a.e.$$

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The most general form of the pointwise ergodic theorem for amenable group actions is due to Lindenstrauss. It relies on averaging over something called a tempered Følner sequence which all amenable groups have. [4]

References

[1] A. del Junco and J. Rosenblatt, *Counterexamples in ergodic theory and number theory*, Mathematische Annalen, 245 (1979), pp. 185–197.

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[4] E. Lindenstrauss, *Pointwise theorems for amenable groups*, Inventiones mathematicae, 146 (2001), pp. 259–295.